Lecture 14

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Applications of Differentiation

Maximum and Minimum values

- Important applications of differentiation are optimization problems, we want to find the optimal way to doing sth. For example

  - Shape of a can that minimizes cost.
  - Maximizing area enclosed by fixed amount of fencing.
  - Maximizing profit.

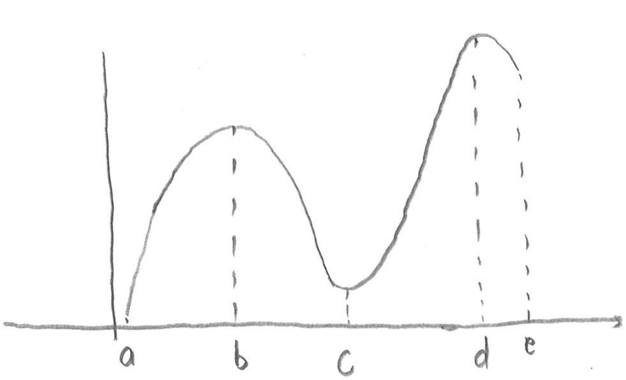
These problems reduce to finding maximum and minimum values of functions.

Defn A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all x in the domain of f. The number $f(c)$ is called the maximum value of f on D.

Similarly a function f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in the domain of f. The number $f(c)$ is called the minimum value of f on D.

The maximum values / minimum values of f are called extreme values of f.

Ex



The function f has absolute maximum at d, and absolute minimum at a.

Def A function has local maximum at c if $f(c) \geq f(x)$, when x is near c (that is it $f(c) \geq f(x)$ for all x in an open interval containing c)

Similarly f has a local minimum at c if $f(c) \leq f(x)$, when x is near c.

Above Ex b is a local maximum, d is a local max

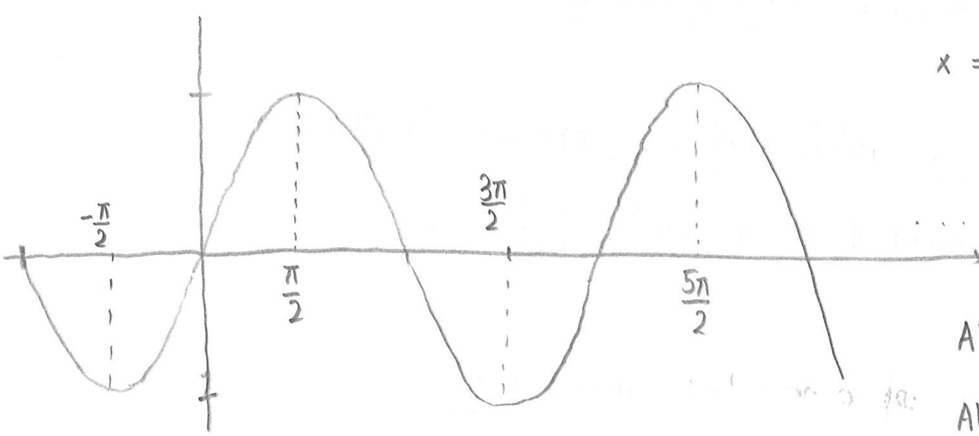
c is a local minimum.

Ex $f(x) = \sin x$

local max at

$x = \frac{\pi}{2} + 2n\pi$, n is an integer

$x = -\frac{\pi}{2} + 2n\pi$, n is an integer.

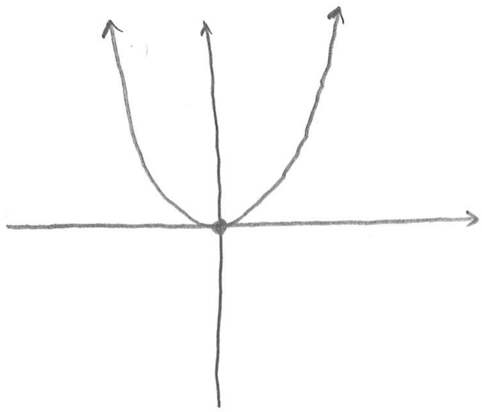


Absolute max value $= 1$

Absolute min value $= -1$

Ex

$f(x) = x^2$



Local min at 0

No Local max

No abs. max

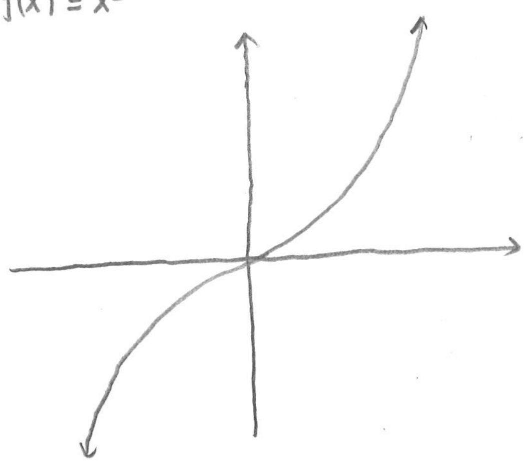
Abs. min at $x = 0$, because

for any real number x, $f(x) \geq 0$

Therefore Abs min value of f is 0.

Ex

$f(x) = x^3$



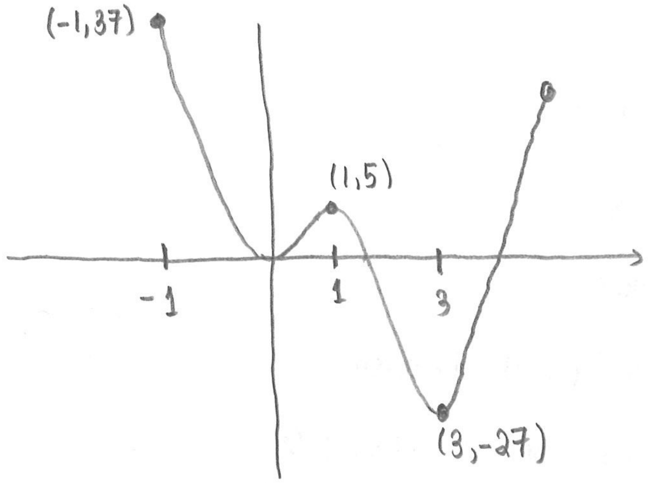
No absolute max

No abs min

No Local max

No Local min

Ex



$f(x) = 3x^4 - 16x^3 + 18x^2$

Abs max is $f(-1) = 37$

Abs min is $f(3) = -27$

local max is $f(1) = 5$

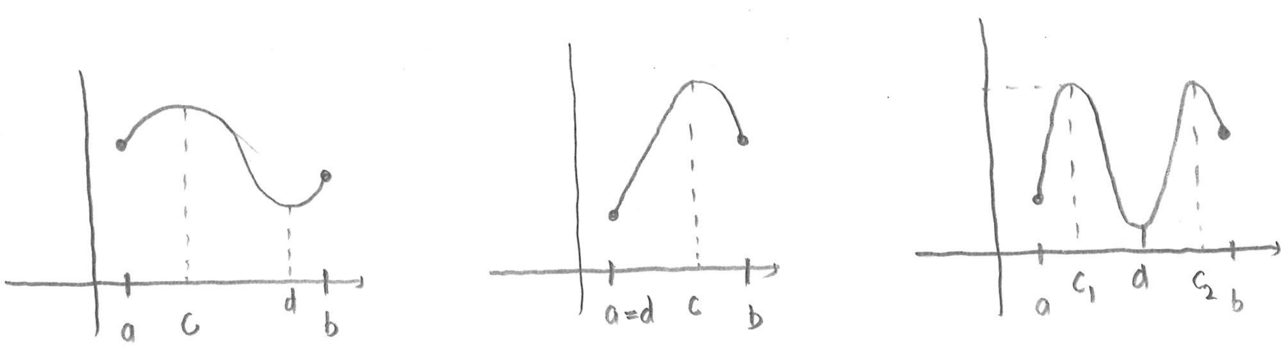
local min are $f(0) = 0$ and $f(3) = -27$

So upto now we have had some functions which have had extreme values, others which did not.

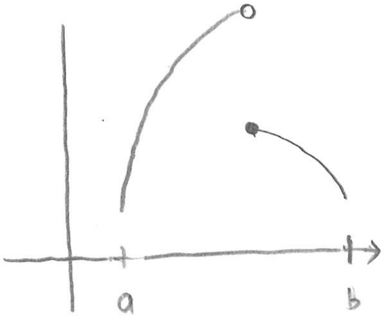
The following Thm gives conditions underwhich a function is guaranteed to possess extreme values

THE EXTREME VALUE THM

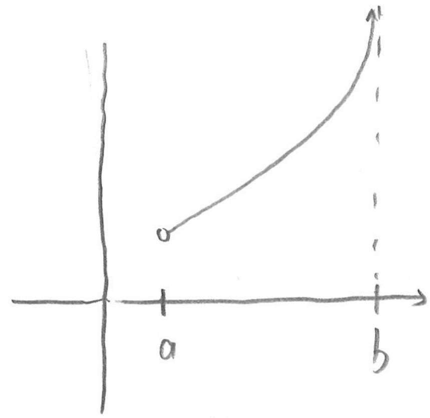
If f is continuous on a closed interval $[a,b]$, then f attains an abs. max. value $f(c)$ and abs. min. value $f(d)$ at some numbers c,d in $[a,b]$.



The continuity of function and closed intervals are important



Not Continuous and hence no abs max



let us consider interval (a, b)

no abs max / no abs min .

If not consider $f(x) = \frac{1}{x-1}$ on $(1,3)$

and no abs min .

- So the Thm tells us that a continuous function on a closed interval has abs. max and abs. min but it doesn't really tell us how.

<u>Fermat's thm.</u>

If f has a local max or local min at c , and if $f'(c)$ exists , then $f'(c) = 0$

<u>IDEA</u>

Then we have that

$f(c) \geq f(c+h)$

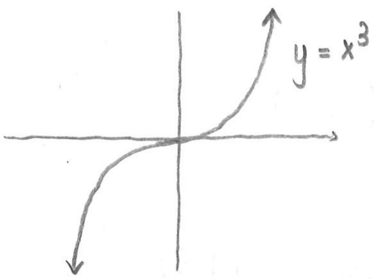
$f(c+h) - f(c) \leq 0$.

Remark 1

The converse is not necessarily true

If $f(x) = x^3$, then $f'(x) = 3x^2$

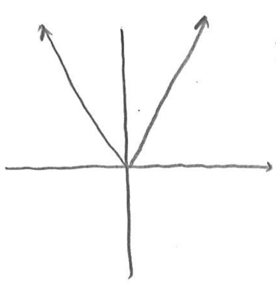
Hence $f'(0) = 0$, but has no maximum or minimum



The fact that $f'(0) = 0$ simply means horizontal tangent line at $(0,0)$. Instead of having a max/min at $(0,0)$, the curve crosses it's horizantal tangent there.

Remark 2

There may be an extreme value where $f'(c)$ does not exist.



$f(x) = |x|$ has it's (local and abs) min value at 0, but $f'(0)$ does not exist.

Defn A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example Find the critical numbers of $f(x) = x^{3/5}(4-x)$

Then, $f(x) = 4x^{3/5} - x^{8/5}$

$$f'(x) = \frac{3}{5} \cdot 4x^{-2/5} - \frac{8}{5}x^{3/5} = \frac{x^{-2/5}}{5}\left[12 - 8x\right]$$

$$= \frac{12-8x}{5x^{2/5}}$$

Then, $f'(x) = 0$ if $12 - 8x = 0 \Rightarrow x = \frac{3}{2}$

Also, $f'(x)$ d.n.e when $x = 0$.

Thus the critical numbers are $x = 0, \frac{3}{2}$.

Restating Fermat's Thm

If f has a local max/min at c, then c is a critical number of f

To find an absolute max or min of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number] or it happens at endpoints of the interval.

Thus the following 3-step procedure always works.

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

1. Find the values of f at the critical numbers of f in (a,b).

2. Find the values of f at the endpoints of the interval.

3. The largest of values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex Find the absolute max and absolute minimum value of $f(x) = x^3 - 3x + 1$ on the interval $[0,3]$?

Soln Since f is a polynomial, it is continuous on $[0,3]$, we can use the Closed Interval method.

$f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$

So, $3x^2 - 3 = 0$

$\Rightarrow 3(x^2 - 1) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

Notice that -1 doesnot lie in the interval $[0,3]$, so we can disregard it.

So we need to evaluate f at the critical number

$f(1) = 1^3 - 3.1 + 1 = -1$

Also need to evaluate f at the endpoints

$f(0) = 0^3 - 3.0 + 1 = 1$

$f(3) = 3^3 - 3.3 + 1 = 27 - 9 + 1 = 19$

Abs max is $f(3) = 19$, Abs min is $f(1) = -1$